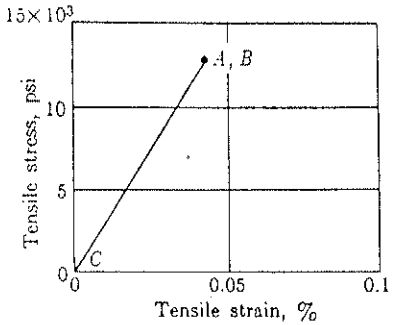
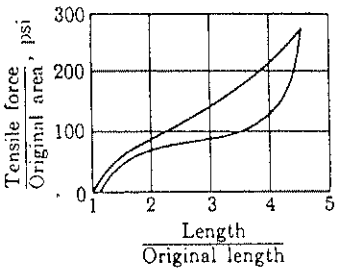
Introduction to Elastic Behavior

- Elastic behavior does not result in permanent deformation upon removal of applied loads
- It is induced primarily by the stretching and rotation of bonds in crystalline and non-crystalline solids
- It may be either linear or non-linear
- It may also be instantaneous or time-dependent

Elastic Deformation of 1020 Steel



Non-linear Deformation of Rubber



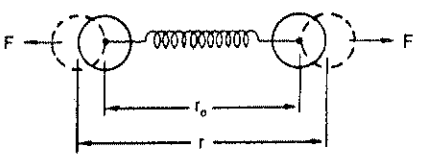
Objectives of this Lecture

- To present an introduction to the basic concepts of elasticity
  - physical basis
  - theoretical basis
- To introduce the anisotropic nature of elastic deformation of crystals
  - single crystals
  - polycrystals

Reasons for Elastic Behavior

- Consider two atoms A and B that are chemically bonded
- The bonds may be either:
  - strong primary bonds (ionic, covalent, metallic)
  - weak secondary bonds (Van der Waal's forces or hydrogen bonds)
- Deformation in early stages controlled by the stretching or rotation of bonds

Stretching of Chemical Bonds Between A and B



Potential Energy and Force Dependence on Atomic Separation

- The potential energy of a bonded system has two components, U_a and U_R:

$$U = U_A + U_R = \frac{-A}{r^m} + \frac{B}{r^n}$$

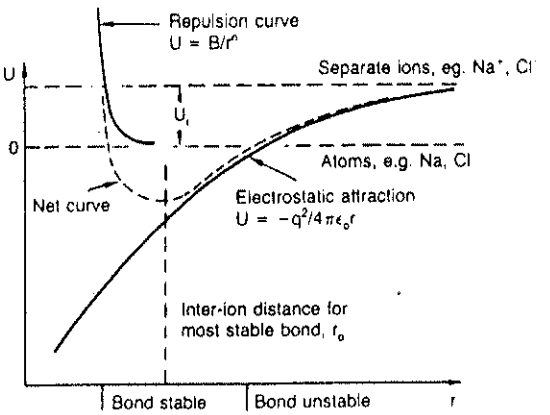
The force is given by the derivative of U

$$F = -\frac{dU}{dr} = -\frac{Am}{r^{m+1}} + \frac{Bn}{r^{n+1}}$$

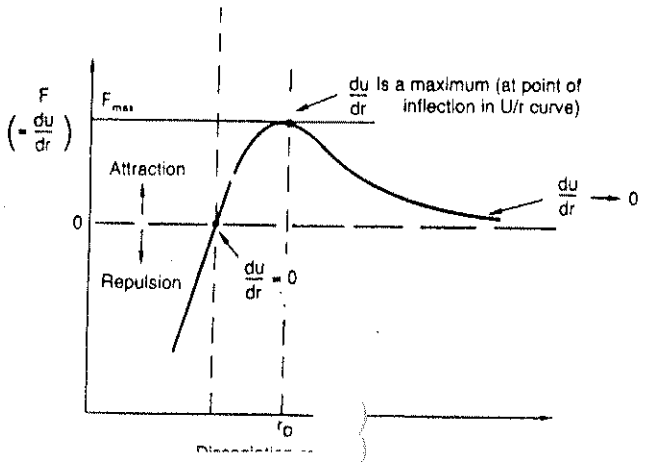
- F = 0 corresponds to the equilibrium separation r_o and binding energy U_o

Schematic of Energy/Force Versus Separation Functions

Relationship Between U and r



Relationship Between F and r



Introduction to Linear Elasticity

- The simple relation between stress and strain was first proposed by Robert Hooke in 1678 (Hooke's law)

$$\sigma = E\varepsilon$$

and

$$\tau = G\gamma$$

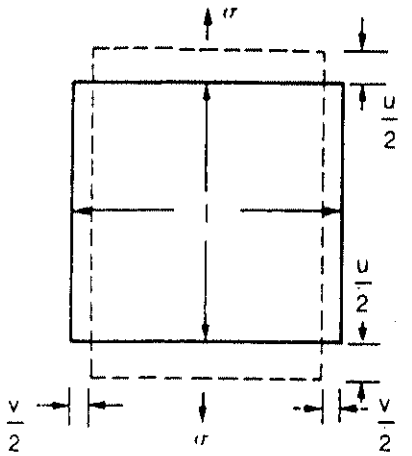
and

$$p = -K\frac{\Delta V}{V} = -K\Delta$$

Introduction to Poisson's Ratio

- Lateral contract generally associated with axial deformation and vice-versa
- This was first recognized by Poisson in the 19th century
- Poisson's ratio is defined as

Schematic of Lateral Contraction

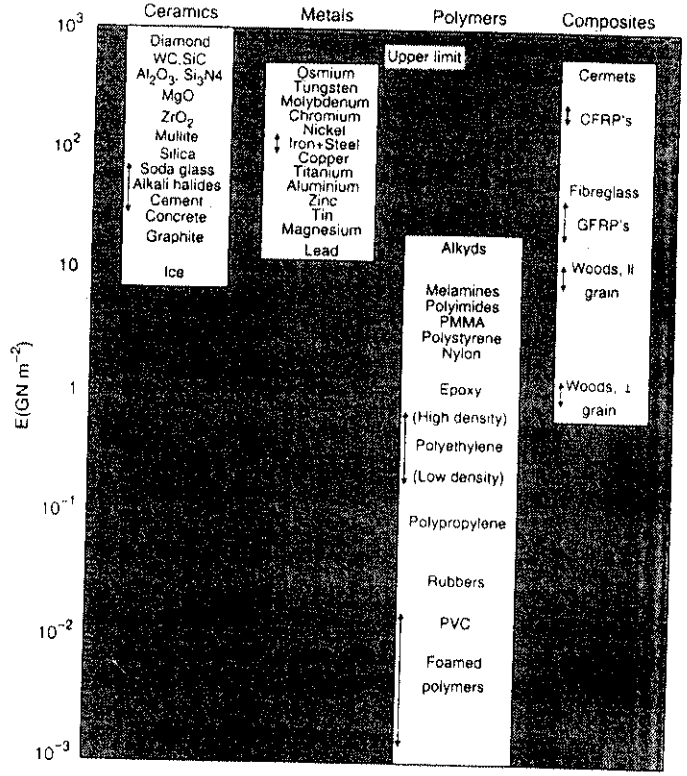


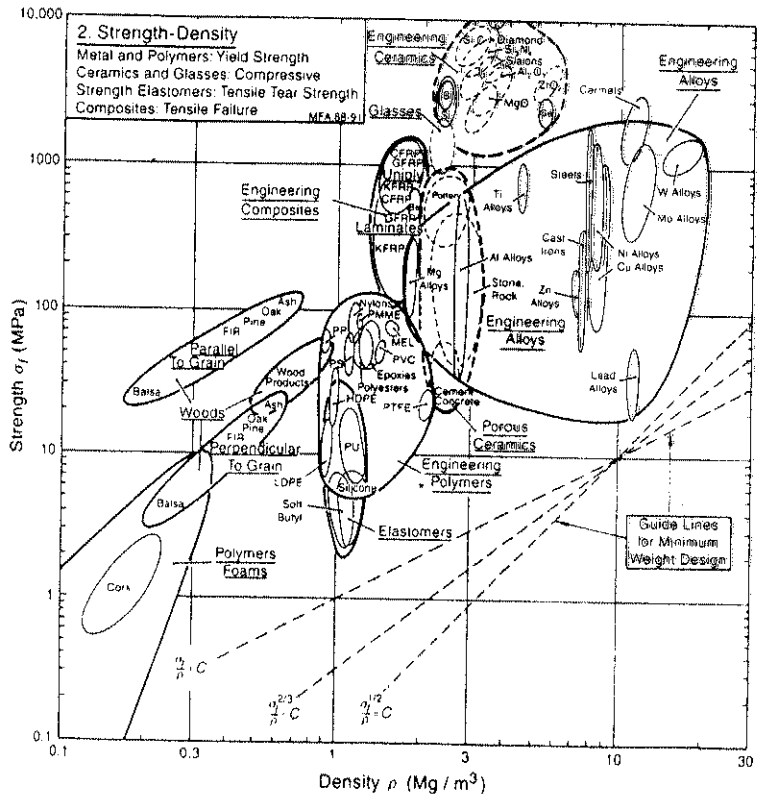
Nominal tensile strain, $\epsilon_n = \frac{u}{l}$

Nominal lateral strain, $\epsilon_n = -\frac{v}{l}$

Poisson's ratio, $\nu = -\frac{\text{lateral strain}}{\text{tensile strain}}$

Typical Elastic Properties for Different Materials





Crystal Symmetry and Anisotropic Deformation

- There appear to be 36 C_{ij} or S_{ij} terms – at first glance
- However, only 21 of these terms are independent
- Number of independent elastic terms decreases with increasing crystal symmetry
    - least symmetric crystals (triclinic crystals) have 21 independent terms
    - orthorhombic have 9 and hexagonal have 5
    - cubic crystals have 3 independent terms

Hooke's Law for Cubic Crystals

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

Introducing the Compliance Tensor

- The earlier expression introduced the elastic stiffness tensor
- The compliance tensor is the inverse of the elastic stiffness tensor
- May also write Hooke's law as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

where the S_{ij} terms are the elastic compliance terms

Introduction to Anisotropic Linear Elasticity

- Crystal deformation is strongly dependent on crystal orientation

- More complete theory must therefore include anisotropy effects

- Can therefore rewrite simple Hooke's law as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$

where the C_{ij} terms are the elastic stiffness terms

Summary of Elastic Stiffness and Compliance Coefficients for Cubic Systems

Material	$(10^{10}$ Pa)			$(10^{-11}$ Pa$^{-1})$		
	c_{11}	c_{12}	c_{44}	s_{11}	s_{12}	s_{44}
Cubic						
Aluminum	10.82	6.13	2.85	1.57	−0.57	3.51
Copper	16.84	12.14	7.54	1.50	−0.63	1.33
Gold	18.60	15.70	4.20	2.33	−1.07	2.38
Iron	23.70	14.10	11.60	0.80	−0.28	0.86
Lithium fluoride	11.2	4.56	6.32	1.16	−0.34	1.58
Magnesium oxide	29.3	9.2	15.5	0.401	−0.096	0.648
Molybdenum[b]	46.0	17.6	11.0	0.28	−0.08	0.91
Nickel	24.65	14.73	12.47	0.73	−0.27	0.80
Sodium chloride[b]	4.87	1.26	1.27	2.29	−0.47	7.85
Spinel ($MgAl_2O_4$)	27.9	15.3	15.3	0.585	−0.208	0.654
Titanium carbide[b]	51.3	10.6	17.8	0.21	−0.036	0.561
Tungsten	50.1	19.8	15.14	0.26	−0.07	0.66
Zinc sulfide	10.79	7.22	4.12	2.0	−0.802	2.43

	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	s_{11}	s_{12}	s_{13}	s_{33}	s_{44}

Estimating Moduli in Different Directions

- The moduli in different directions may be estimated from the stiffness coefficients

- For cubic crystals, the Young's modulus in a given hkl direction is given by

$$\frac{1}{E_{hkl}} = S_{11} - 2\left[(S_{11} - S_{12}) - \frac{S_{44}}{2}\right](l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_2^2)$$

where l_1, l_2 and l_3 are the direction cosines of the angle between the direction and the (x,y,z) axes, respectively

Summary of Elastic Direction Cosines and Anisotropy Ratios

Direction Cosines

Direction	l_1	l_2	l_3
⟨100⟩	1	0	0
⟨110⟩	$1/\sqrt{2}$	$1/\sqrt{2}$	0
⟨111⟩	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

Anisotropy Ratios

Metal	Relative Degree of Anisotropy $\left[\frac{2(s_{11} - s_{12})}{s_{44}}\right]$	E_{111} (10^6 psi)	E_{100} (10^6 psi)	$\left[\frac{E_{111}}{E_{100}}\right]$
Aluminum	1.219	11.0	9.2	1.19
Copper	3.203	27.7	9.7	2.87
Gold	2.857	16.9	6.2	2.72
Iron	2.512	39.6	18.1	2.18
Magnesium oxide	1.534	50.8	36.2	1.404
Spinel ($MgAl_2O_4$)	2.425	52.9	24.8	2.133
Titanium carbide	0.877	62.2	69.1	0.901
Tungsten	1.000	55.8	55.8	1.00

Estimating the Young's Moduli of Polycrystalline Materials

- Simplest approaches simply use a rule of mixture approach (limited accuracy)

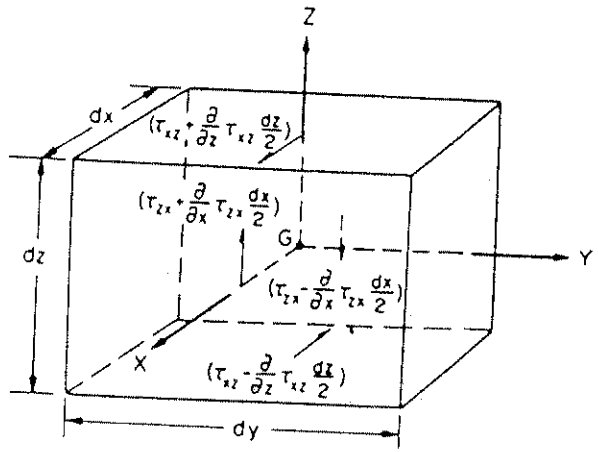
$$E = \sum_{i=1}^{n} v_i E_i$$

- Bounds in the elastic properties can also be estimated using statistical mechanics or composite theories
- Energy-based approaches provide the best results
- Isotropic elasticity is a true idealization – only true in a few cases e.g. tungsten

Introduction to the Equilibrium Conditions

- The relationships between stress and strain may be determined by the conditions for equilibrium of a cubic element
- Equilibrium can be achieved by considering all the forces that are needed to keep the element suspended in space
- These consist of normal/shear stresses and body forces such as gravitational or centrifugal forces

Equilibrium in the Cartesian Coordinate System



Equilibrium in the Cartesian Coordinate System

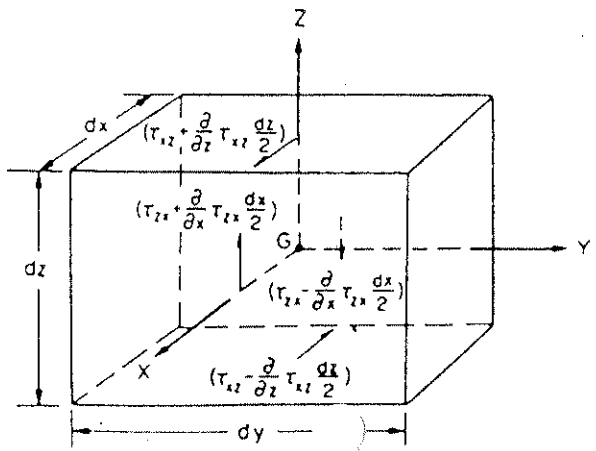
- Considering the gradients in stress across the cube and force equilibrium in the (x,y,z) directions gives:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = \rho \frac{\partial^2 x}{\partial t^2}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = \rho \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = \rho \frac{\partial^2 z}{\partial t^2}$$

State of Stress on an Element



Solving for Stress in Equilibrium

- There are 6 stress components and 3 equilibrium equations
- Hence, solving for stress terms requires stress-strain relationships in most problems in linear elasticity
- This can be achieved by
    - simplifying or idealizing the stress states (plane stress versus plane strain)
    - or by solving full 3-D elasticity problem

Equilibrium in Spherical Coordinate System

- Similarly equilibrium conditions may be derived for spherical coordinate system
- Summing forces components in the r, θ and ϕ directions and neglecting body forces gives

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \phi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \tau_{r\phi}\cot\phi}{r} + F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{3\tau_{\phi\theta}}{r}\cot\phi + F_\theta = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{\phi\phi} - \sigma_{\theta\theta}}{r}\cot\phi + \frac{3\tau_{r\phi}}{r} + F_\phi = 0$$

Equilibrium for Cylindrical and Spherical Coordinate Systems

- For bodies with cylindrical and spherical symmetry, cylindrical and spherical coordinate systems may be used

- Similarly, the equilibrium equations may be derived from force equilibrium

- For cylindrical bodies equilibrium conditions are given by

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{zr}}{r} = 0$$

Equilibrium in Cylindrical Coordinate System

Equilibrium for Stationary Bodies in the Absence of Body Forces

- In the absence of body forces $F_x = F_y = F_z = 0$
- For stationary bodies in equilibrium – acceleration terms

$$\rho\frac{\partial 2_x}{dt^2} = \rho\frac{\partial 2_y}{\partial dt^2} = \rho^2\frac{\partial z}{\partial dt^2} = 0$$

- Hence, equilibrium equations in Cartesian coordinate system become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

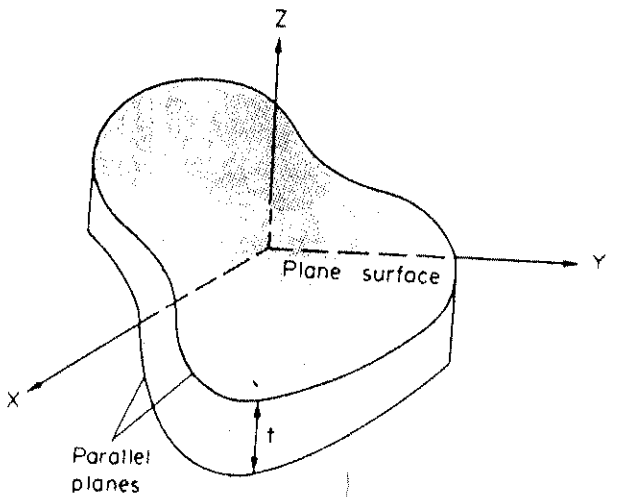
$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Plane Stress and Plane Strain Conditions

- In many problems, it is possible to achieve considerable simplification by assuming biaxial stress (plane stress) or biaxial strain (plane strain) conditions
- Plane stress conditions often apply to problems where the thickness is small, while plane strain often applies to thick sections
- However, there are examples in which constraint can lead to plane strain conditions

Schematic of a Plane Element



Introduction to Plane Stress Problems

- In plane stress problems, all the z components of stress are zero

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

- The value of σ_{zz} may also be reduced to zero by imposing an equal stress of opposite sign in the z-direction

- The equations of equilibrium reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0$$

Equilibrium Conditions Under Plane Strain Conditions

- Under plane strain conditions the strains in the z-direction are zero

$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

- The equilibrium equations are now given by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0$$

$$\frac{\partial \sigma_{zz}}{\partial z} + F_z = 0$$

The Relationships Between Stress and Strain Under Plane Stress Conditions

- Under plane stress conditions, the relationships between stress and strain are given by (Hooke's Law):

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

The Relationships Between Strain and Stress Under Plane Strain Conditions

- Under plane strain conditions, the relationships between strain and stress are given by

$$\varepsilon_{xx} = \frac{(1-\nu^2)}{E}\left[\sigma_{xx} - \frac{\nu}{(1-\nu)}\sigma_{yy}\right]$$

$$\varepsilon_{yy} = \frac{(1-\nu^2)}{E}\left[\sigma_{yy} - \frac{\nu}{(1-\nu)}\sigma_{xx}\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Plane Stress/Strain Relationships Between Stress and Strain

- For plane stress, we have

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{yy} + \nu\varepsilon_{xx}\right]$$

$$\tau_{xy} = G\gamma_{xy}$$

- For plane strain, we have

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{xx} + \frac{\nu}{(1-\nu)}\varepsilon_{yy}\right]$$

$$\sigma_{yy} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{yy} + \frac{\nu}{(1-\nu)}\varepsilon_{xx}\right]$$

$$\tau_{xy} = G\gamma_{xy}$$

Generalized Three-Dimensional State of Stress

- Thermal stresses may be included on the equilibrium equations by treating them as body forces

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \alpha_x \Delta T = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \alpha_y \Delta T = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \alpha_z \Delta T = 0$$

- Similar expressions may be written for cylindrical and spherical coordinate systems
- However, equilibrium equations cannot be solved without satisfying compatibility conditions

Compatibility Conditions and Stress Functions

- The compatibility conditions are needed to ensure single-valved displacements
- These are derived in most tests on elasticity
- They are given by equations of the form (for subscripts i and j between 1 and 3):

$$\frac{\partial^2 \varepsilon_{ii}}{\partial x_j^2} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i^2} = \frac{\partial^2 \gamma_{ij}}{\partial x_i \partial x_j}$$

$$\frac{2\partial^2 \varepsilon_{ji}}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_i}\left(-\frac{\partial \gamma_{jk}}{\partial x_i} + \frac{\partial \gamma_{ki}}{\partial x_j} + \frac{\partial \gamma_{ij}}{\partial x_k} \right)$$

Solving Problems in Linear Elasticity

- Solving problems in linear elasticity can be difficult due to number and type of equations that have to be satisfied
- It is common to use trial and error methods to obtain solutions
- One class of trial functions developed by Sir George Airy is known as Airy functions

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}$$

$$\tau_{xy} = \frac{-\partial^2 \chi}{\partial x \partial y}$$

The Compatibility Condition in Terms of the Airy Stress Function

- In Cartesian coordinates, the compatibility condition may be expressed in terms of the Airy stress function

$$\frac{\partial^4 \chi}{\partial x^4} + \frac{2\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0$$

- For stress fields with polar symmetry, the Airy stress functions are given by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \chi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right)$$

Satisfying the Compatibility Conditions with the Airy Stress Functions

- The compatibility conditions are satisfied with the Airy stress function when Laplace's equation and the biharmormic equation are satisfied

$$\nabla^2 c = 0$$

and

$$\nabla^4 c = 0$$

where ∇^2 is the operator given by

$$\nabla^4 c = 0$$

Summary and Concluding Remarks

- Introduction to elasticity theory presented along with physical basis for elasticity
- Physical basis for elasticity is the stretching and rotation of bonds
- Hooke's law presented for 1D and 3D
- Basic introduction to anisotropic linear elasticity was presented for single crystals and polycrystals
- Equilibrium expressions described along with methods for solving elasticity problems